

TWINING CHARACTERS AND ORBIT LIE ALGEBRAS [†]

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Abstract.

We associate to outer automorphisms of generalized Kac–Moody algebras generalized character-valued indices, the *twining characters*. A character formula for twining characters is derived which shows that they coincide with the ordinary characters of some other generalized Kac–Moody algebra, the so-called *orbit Lie algebra*. Some applications to problems in conformal field theory, algebraic geometry and the theory of sporadic simple groups are sketched.

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1 Generalized Kac–Moody algebras

Generalized Kac–Moody algebras constitute a class of Lie algebras which comprise many Lie algebras that describe symmetries in physical systems. In particular, they include the finite-dimensional simple Lie algebras (i.e. the four series of classical Lie algebras and the five exceptional simple Lie algebras), twisted and untwisted affine Lie algebras (i.e. centrally extended loop algebras), hyperbolic Lie algebras (such as E_{10}), as well as the Monster Lie algebra and its various relatives. Moreover, any chiral algebra of a conformal field theory, i.e. any vertex operator algebra, gives rise to a generalized Kac–Moody algebra.

Any finite-dimensional simple Lie algebra is generated (as a Lie algebra) by several copies of $\mathfrak{sl}(2)$, one copy for each simple root. This is also true for ordinary Kac–Moody algebras; a generalized Kac–Moody algebra, however, is generated by copies of $\mathfrak{sl}(2)$ and of the Heisenberg algebra which has a basis $\{e, f, K\}$ with K a central element and non-trivial Lie bracket $[e, f] = K$. Nonetheless, generalized Kac–Moody algebras can still be characterized by a square matrix, the Cartan matrix $A = (a_{ij})_{i,j \in I}$. The index set I can either be finite, $I = \{1, 2, \dots, n\}$, or countably infinite, $I = \mathbf{Z}_+$. In case of finite-dimensional simple Lie algebras the entries of A are integers; here we allow for real entries, but still we keep the following properties of A :

- (i) $a_{ij} \leq 0$ if $i \neq j$; (ii) $\frac{2a_{ij}}{a_{ii}} \in \mathbf{Z}$ if $a_{ii} > 0$;
 - (iii) if $a_{ij} = 0$, then $a_{ji} = 0$;
 - (iv) there exists a diagonal matrix $D = \text{diag}(\epsilon_1, \dots, \epsilon_n)$, with ϵ_i a positive real number for all i , such that DA is symmetric.
- (1.1)

From these data the generalized Kac–Moody algebra is constructed by the same procedure that is used to construct a finite-dimensional simple Lie algebra from its Cartan matrix. One starts with an abelian Lie algebra \mathfrak{g}_0 of dimension greater or equal to n and fixes n linearly independent elements h_1, \dots, h_n of \mathfrak{g}_0 and n linear forms α_j on \mathfrak{g}_0 , the simple roots, such that $\alpha_j(h_i) = a_{ij}$. The generalized Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ with Cartan matrix A and Cartan subalgebra \mathfrak{g}_0 is then the Lie algebra generated by $e_i, f_i, i \in I$, and \mathfrak{g}_0 , modulo the relations

$$\begin{aligned}
 [e_i, f_j] &= \delta_{ij} h_i, & [h, e_i] &= \alpha_i(h) e_i, & [h, f_i] &= -\alpha_i(h) f_i, \\
 (\text{ad } e_i)^{1-2a_{ij}/a_{ii}} e_j &= 0 = (\text{ad } f_i)^{1-2a_{ij}/a_{ii}} f_j & \text{ if } a_{ii} > 0, \\
 [e_i, e_j] &= 0 = [f_i, f_j] & \text{ if } a_{ij} = 0.
 \end{aligned}$$
(1.2)

2 Automorphisms of generalized Kac–Moody algebras

We now turn to the main object of our interest: a class of outer automorphisms of generalized Kac–Moody algebras and some new structures associated to them. Such automorphisms occur naturally in many physical applications; for some examples see the last section and [5]. We start with a permutation $\hat{\omega}$ of finite order of the index set I which leaves the Cartan matrix A invariant:

$$a_{\hat{\omega}i, \hat{\omega}j} = a_{i,j}. \quad (2.3)$$

If the generalized Kac–Moody algebra has a Dynkin diagram, $\hat{\omega}$ corresponds to a symmetry of the Dynkin diagram. Such a permutation $\hat{\omega}$ induces an automorphism $\omega: \mathfrak{g} \rightarrow \mathfrak{g}$ of the Lie algebra \mathfrak{g} ,

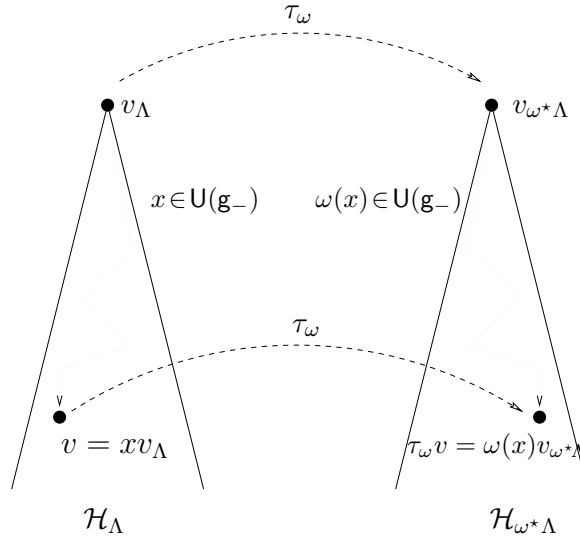
which is defined by its action on the generators e_i , f_i and h_i :¹

$$\omega(e_i) := e_{\dot{\omega}i}, \quad \omega(f_i) := f_{\dot{\omega}i}, \quad \omega(h_i) := h_{\dot{\omega}i}. \quad (2.4)$$

The automorphism ω gives rise to a map $\tau_\omega : \mathcal{H}_\Lambda \rightarrow \mathcal{H}_{\omega^*\Lambda}$ of integrable highest weight modules (and analogously also for Verma modules) which ‘ ω -twines’ the action of \mathfrak{g} :

$$\tau_\omega(R_\Lambda(x) \cdot v) = R_{\omega^*\Lambda}(\omega(x)) \cdot \tau_\omega(v), \quad (2.5)$$

and maps a fixed highest weight vector v_Λ of \mathcal{H}_Λ to a highest weight vector $v_{\omega^*\Lambda}$ of $\mathcal{H}_{\omega^*\Lambda}$. Analogous maps can also be defined for Verma modules; they are compatible with the submodule structure of the Verma modules and hence also with their null vector structures. Pictorially, the map τ_ω can be represented as follows:



3 Fixed points

Note that τ_ω is generically a linear map from a module \mathcal{H}_Λ to a different module $\mathcal{H}_{\omega^*\Lambda}$. A particularly interesting situation occurs when the highest weight Λ is *symmetric*, i.e. satisfies $\omega^*(\Lambda) = \Lambda$. In this case the highest weight module \mathcal{H}_Λ is called a *fixed point* of ω and τ_ω is an endomorphism of the vector space underlying the module \mathcal{H}_Λ . To keep track of some properties of the endomorphism τ_ω , we introduce a character-like object, the *twining character* $\chi_\Lambda^{[\omega]}$:

$$\chi_\Lambda^{[\omega]}(h) = \text{tr}_{\mathcal{H}_\Lambda} \tau_\omega e^{2\pi i R_\Lambda(h)} \quad \text{for } h \in \mathfrak{g}_0. \quad (3.6)$$

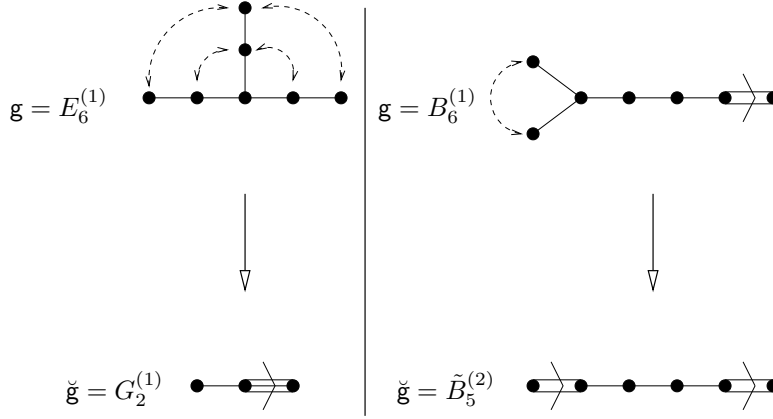
Like ordinary characters the twining characters are (formal) functions on the Cartan subalgebra. They are the generating functions of the trace of τ_ω on the weight spaces, i.e. they are generalized (since ω

¹ In general, one also must describe the action of ω on the derivations of the generalized Kac–Moody algebra. For more details see section 3 of [2].

does not necessarily have order two) character-valued indices. Clearly, for the trivial automorphism $\omega = \mathbf{1}$ we recover the ordinary character of the module \mathcal{H}_Λ .

It is a rather surprising result [2, 1] that the twining character is essentially identical to the character of some other generalized Kac–Moody algebra $\check{\mathfrak{g}}$, called the *orbit Lie algebra*. This result makes the twining characters explicitly computable. In particular, it implies that the coefficients in the expansion of the twining character are not arbitrary complex numbers, but non-negative integers, and hence justifies a posteriori the name *twining character*.

The orbit Lie algebra corresponding to \mathfrak{g} and its diagram automorphism ω is obtained by a simple prescription which corresponds to folding the Dynkin diagram of \mathfrak{g} according to the action of ω . Pictorially we have e.g.:



In formulæ, the orbit Lie algebra is described as follows. Denote by \hat{I} a set of representatives in I for each ω -orbit. The Cartan matrix of the orbit Lie algebra is then labelled by the subset

$$\check{I} := \{i \in \hat{I} \mid \sum_{l=0}^{N_i-1} a_{i,\omega^l i} \leq 0 \implies \sum_{l=0}^{N_i-1} a_{i,\omega^l i} = a_{ii}\} \quad (3.7)$$

of \hat{I} . For each $i \in \hat{I}$ we define the number

$$s_i := \begin{cases} a_{ii} / \sum_{l=0}^{N_i-1} a_{i,\omega^l i} & \text{if } i \in \check{I} \text{ and } a_{ii} \neq 0, \\ 1 & \text{otherwise,} \end{cases} \quad (3.8)$$

which is either 1 or 2. The Cartan matrix $\hat{A} = (\hat{a}_{ij})_{i,j \in \check{I}}$ of the orbit Lie algebra is then defined by summing over one index of the Cartan matrix of \mathfrak{g} :

$$\hat{a}_{ij} := s_j \sum_{l=0}^{N_j-1} a_{i,\omega^l j}. \quad (3.9)$$

We emphasize that $\check{\mathfrak{g}}$ is not constructed as a subalgebra of \mathfrak{g} ; in particular the orbit Lie algebra is in general not isomorphic to the subalgebra of \mathfrak{g} that stays fixed under ω . One can also show that the algebra \mathfrak{g} and the orbit Lie algebra $\check{\mathfrak{g}}$ are ² of the same type (i.e. simple, affine, indefinite); the orbit Lie algebra of hyperbolic Lie algebras is hyperbolic as well. A particularly interesting situation

² except for the order- N automorphisms of $\mathfrak{sl}(N)$

is illustrated on the right hand side of the figure: for certain orbit Lie algebras, one obtains the twisted affine Lie algebras $\tilde{B}_n^{(2)}$ which are the only twisted affine Lie algebras for which the characters have good modular transformation properties. This had been predicted [6] on the basis of level-rank dualities of $N = 2$ superconformal coset theories.

We conjecture that when the original Lie algebra \mathfrak{g} is an untwisted affine Lie algebra and the symmetry of the Dynkin diagram corresponds to the action of a simple current,³ then the modular matrix S of the orbit Lie algebra $\check{\mathfrak{g}}$ describes the modular transformation properties of the chiral blocks for the one-point function of the relevant simple current on the torus. This would provide a conceptual explanation of the observation that in these cases the twining characters of \mathfrak{g} span again a unitary representation of $SL(2, \mathbf{Z})$. For some further evidence for this conjecture, see [5].

4 Sketch of the proof

We will now outline the main ideas which enter in the proof of a character formula for the twining character; for the full details we refer the reader to the original publications [2, 1]. The first step is to show that the subgroup \hat{W} of the Weyl group W of \mathfrak{g} that consists of those elements of W whose action on the weight space of \mathfrak{g} commutes with ω^\star is isomorphic to the Weyl group \check{W} of the orbit Lie algebra $\check{\mathfrak{g}}$. This part of the proof also provides new insight in the structure of Coxeter groups.

The idea is to show that \hat{W} plays for the twining characters the role the full Weyl group plays for the ordinary characters: First, the twining characters of Verma modules are \hat{W} -odd. More precisely, let $\mathcal{V}_\Lambda^{(\omega)}$ denote the twining character of the Verma module; then $\mathcal{V}^{(\omega)} := e^{-\rho-\Lambda} \mathcal{V}_\Lambda^{(\omega)}$ (which does not depend on Λ) is antisymmetric under \hat{W} , $\hat{w}(\mathcal{V}^{(\omega)}) = \hat{\epsilon}(\hat{w}) \mathcal{V}^{(\omega)}$. Here $\hat{\epsilon}$ is the sign function associated to \hat{W} as a Coxeter group, and not the restriction of the sign function of W . The twining characters $\chi_\Lambda^{[\omega]}$ of the irreducible modules, on the other hand, are symmetric under the action of \hat{W} :

$$\hat{w}(\chi_\Lambda^{[\omega]}) = \chi_\Lambda^{[\omega]}. \quad (4.10)$$

We can now generalize the arguments used in the proof of the Weyl-Kac-Borcherds character formula of ordinary characters and implement the symmetry properties under \hat{W} to derive an explicit character formula for the twining characters:

$$\chi_\Lambda^{[\omega]} = \frac{\sum_{\hat{w} \in \hat{W}} \hat{\epsilon}(\hat{w}) \hat{w}(S_\Lambda^\omega)}{\sum_{\hat{w} \in \hat{W}} \hat{\epsilon}(\hat{w}) \hat{w}(S_0^\omega)}, \quad (4.11)$$

where

$$S_\Lambda^\omega = e^{\Lambda+\rho} \sum \hat{\epsilon}(\beta) e^{-\beta} \quad (4.12)$$

and $\hat{\epsilon}(\beta) = (-1)^n$ if β is the symmetric sum of n pairwise orthogonal imaginary simple roots, all orthogonal to Λ , and $\hat{\epsilon}(\beta) = 0$ otherwise. Comparison with the character formula for the orbit Lie algebra then proves the claim.

³ Dynkin diagram symmetries coming from simple currents [7] are elements of the unique maximal abelian normal subgroup of the diagram automorphisms. This subgroup is isomorphic to the center of the universal covering Lie group that has the horizontal subalgebra $\check{\mathfrak{g}} \subset \mathfrak{g}$ as its Lie algebra.

5 Applications and conclusions

Orbit Lie algebras and twining characters are novel structures in the representation theory of generalized Kac–Moody algebras. They already found several applications in two-dimensional conformal field theory, e.g. in the solution of the problem of field identification fixed points in coset conformal field theories [3] and in fixed point resolution for integer spin simple current (‘ D -type’) modular invariants of conformal field theories [4]. More details can be found in the original publications and in [5].

In the case of WZW theories, integer spin modular invariants describe the WZW theory on a non-simply connected group manifold. In general, twining characters and orbit Lie algebras play a role as soon as one considers non-simply connected Lie groups. In particular, they allow to describe Chern–Simons theories based on such groups [8] and they give a Verlinde formula also for non-simply connected groups [4]. Other applications concern the biggest sporadic simple group, the monster group, which acts on a generalized Kac–Moody algebra, the monster Lie algebra, by outer automorphisms (recently, this group has been proposed as the symmetry underlying $N = 2$ string theory). We expect that orbit Lie algebras and twining characters will be a useful tool in this situation as well. Finally, twining characters of finite-dimensional simple Lie algebras are closely related to the characters of non-connected finite-dimensional simple Lie groups (the latter have e.g. been studied in [9]).

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